Differentialrechnung

Funktionen

$$\mathbf{f} : (D \subseteq \mathbb{R}^n) \to \mathbb{R}^m \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \to \mathbf{f}(\mathbf{x}) = \begin{pmatrix} f_1(\mathbf{x}) \\ \vdots \\ f_m(\mathbf{x}) \end{pmatrix} \qquad i \in [1 \dots m] \qquad j \in [1 \dots n]$$

Jacobi-Matrix $\mathbf{J}(\mathbf{f}(\mathbf{x})) = \frac{\partial \mathbf{f}}{\partial \mathbf{x}^{\mathrm{T}}}(\mathbf{x}) = \left(j_{ij}(\mathbf{x})\right) = \left(\frac{\partial f_i}{\partial x_j}(\mathbf{x})\right) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}) & \dots & \frac{\partial f_1}{\partial x_n}(\mathbf{x}) \\ \dots & \frac{\partial f_i}{\partial x_j}(\mathbf{x}) & \dots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{x}) & \dots & \frac{\partial f_m}{\partial x_n}(\mathbf{x}) \end{pmatrix}$

Grenzwert $\lim(\mathbf{f}(\mathbf{x})) = \sum_{i=1}^{m} \lim(f_i(\mathbf{x}))\,\mathbf{e}_i$

Ableitung $\frac{\partial \mathbf{f}}{\partial x_j}(\mathbf{x}) = \sum_{i=1}^{m} j_{ij}\mathbf{e}_i$

Differential $\mathrm{d}\mathbf{f}(\mathbf{x}) = \sum_{i=1}^{m}\sum_{j=1}^{n} j_{ij}\mathrm{d}x_j\mathbf{e}_i \qquad \mathrm{d}^k\mathbf{f}(\mathbf{x}) = \sum_{i=1}^{m}\left(\sum_{j=1}^{n} \mathrm{d}x_j \frac{\partial}{\partial x_j}\right)^k f_i(\mathbf{x})\mathbf{e}_i$

Produktregel	Quotientenregel	Kettenregel	Partielle Integration
$\frac{\mathrm{d}}{\mathrm{d}t}(uv) = \dot{u}v + u\dot{v}$	$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{u}{v}\right) = \frac{\dot{u}v - u\dot{v}}{v^2}$	$\frac{\mathrm{d}}{\mathrm{d}t}(u(v)) = \frac{\mathrm{d}u}{\mathrm{d}v}\frac{\mathrm{d}v}{\mathrm{d}t}$	$\int_a^b \dot{u}v\,\mathrm{d}t = [uv]_a^b - \int_a^b u\dot{v}\,\mathrm{d}t$

Nabla

Gradient $\operatorname{grad}(f_i(\mathbf{x})) = \nabla f_i = \sum_{j=1}^{n} j_{ij}\mathbf{e}_j = \lim_{V\to 0}\left(\frac{1}{V}\oint\!\!\oint_{\mathbf{A}=\delta V} f_i \mathrm{d}\mathbf{A}\right)$

$m = n$:

Divergenz $\operatorname{div}(\mathbf{f}(\mathbf{x})) = \nabla \cdot \mathbf{f} = \sum_{i=1}^{m} j_{ii} = \lim_{V\to 0}\left(\frac{1}{V}\oint\!\!\oint_{\mathbf{A}=\delta V} \mathbf{f}\cdot\mathrm{d}\mathbf{A}\right)$

Laplace $\Delta\mathbf{f}(\mathbf{x}) = \nabla^2\mathbf{f} = \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{\partial j_{ij}}{\partial x_j}\mathbf{e}_i$

$m = n = 3$:

Rotation $\operatorname{rot}(\mathbf{f}(\mathbf{x})) = \nabla \times \mathbf{f} = \begin{pmatrix} j_{32} - j_{23} \\ j_{13} - j_{31} \\ j_{21} - j_{12} \end{pmatrix} = \lim_{V\to 0}\left(\frac{1}{V}\oint\!\!\oint_{\mathbf{A}=\delta V} \mathrm{d}\mathbf{A}\times\mathbf{f}\right) = \frac{\mathbf{A}}{|\mathbf{A}|}\lim_{\mathbf{A}\to\mathbf{0}}\left(\frac{1}{|\mathbf{A}|}\oint_{\mathbf{s}=\delta\mathbf{A}} \mathbf{f}\cdot\mathrm{d}\mathbf{s}\right)$

Identitäten $\nabla \times (\nabla f) = \mathbf{0} \qquad \nabla \cdot (\nabla \times \mathbf{f}) = 0$

Integralsätze

Differenz von Funktionswerten	Integral Vektorfeld über Kurve	Fluß Vektorfeld durch Fläche	Volumentintegral
$f(\mathbf{b}) - f(\mathbf{a})$	$\int_{\mathbf{s}} \mathbf{v}(\mathbf{x})\cdot\mathrm{d}\mathbf{x}$	$\iint_{\mathbf{A}} \mathbf{w}(\mathbf{x})\cdot\mathrm{d}\mathbf{A}$	$\iiint_V g(\mathbf{x})\mathrm{d}V$

Hauptsatz	Stokes	Gauß
$f(\mathbf{b}) - f(\mathbf{a}) = \int_{\mathbf{s}=\mathbf{a}}^{\mathbf{b}} \mathbf{v}(\mathbf{x})\cdot\mathrm{d}\mathbf{x}$	$\oint_{\mathbf{s}=\delta\mathbf{A}} \mathbf{v}(\mathbf{x})\cdot\mathrm{d}\mathbf{x} = \iint_{\mathbf{A}} \mathbf{w}(\mathbf{x})\cdot\mathrm{d}\mathbf{A}$	$\oint\!\!\oint_{\mathbf{A}=\delta V} \mathbf{w}(\mathbf{x})\cdot\mathrm{d}\mathbf{A} = \iiint_V g(\mathbf{x})\mathrm{d}V$
Bedingung	Bedingung	Bedingung
$\nabla \times \mathbf{v} = \mathbf{0} \quad \Leftrightarrow \quad \mathbf{v} = \nabla f$	$\nabla \cdot \mathbf{w} = 0 \quad \Leftrightarrow \quad \mathbf{w} = \nabla \times \mathbf{v}$	$g = \nabla \cdot \mathbf{w}$

Koordinatensysteme

	Karthesisch	Zylinder	Kugel
Vektoren	$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-\infty\dots\infty\\-\infty\dots\infty\\-\infty\dots\infty\end{pmatrix}$	$\begin{pmatrix}\rho\cos(\varphi)\\\rho\sin(\varphi)\\z\end{pmatrix}$	$\begin{pmatrix}r\cos(\varphi)\sin(\vartheta)\\r\sin(\varphi)\sin(\vartheta)\\r\cos(\vartheta)\end{pmatrix}$
	$\begin{pmatrix}\sqrt{x^2+y^2}\\\arctan\left(\frac{y}{x}\right)\\z\end{pmatrix}$	$\begin{pmatrix}\rho\\\varphi\\z\end{pmatrix}=\begin{pmatrix}0\dots\infty\\0\dots2\pi\\-\infty\dots\infty\end{pmatrix}$	$\begin{pmatrix}r\sin(\vartheta)\\\varphi\\r\cos(\vartheta)\end{pmatrix}$
	$\begin{pmatrix}\sqrt{x^2+y^2+z^2}\\\arctan\left(\frac{y}{x}\right)\\\arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right)\end{pmatrix}$	$\begin{pmatrix}\sqrt{\rho^2+z^2}\\\varphi\\\arctan\left(\frac{\rho}{z}\right)\end{pmatrix}$	$\begin{pmatrix}r\\\varphi\\\vartheta\end{pmatrix}=\begin{pmatrix}0\dots\infty\\0\dots2\pi\\0\dots\pi\end{pmatrix}$
Wegelement d**s**	$\begin{pmatrix}\mathrm{d}x\\\mathrm{d}y\\\mathrm{d}z\end{pmatrix}$	$\begin{pmatrix}\mathrm{d}\rho\\\rho\mathrm{d}\varphi\\\mathrm{d}z\end{pmatrix}$	$\begin{pmatrix}\mathrm{d}r\\r\sin(\vartheta)\mathrm{d}\varphi\\r\mathrm{d}\vartheta\end{pmatrix}$
Einheitsvektoren	$\begin{pmatrix}\mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}\cos(\varphi)&-\sin(\varphi)&0\\\sin(\varphi)&\cos(\varphi)&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}\cos(\varphi)\sin(\vartheta)&-\sin(\varphi)&\cos(\varphi)\cos(\vartheta)\\\sin(\varphi)\sin(\vartheta)&\cos(\varphi)&\sin(\varphi)\cos(\vartheta)\\\cos(\vartheta)&0&-\sin(\vartheta)\end{pmatrix}$
	$\begin{pmatrix}\cos(\varphi)&\sin(\varphi)&0\\-\sin(\varphi)&\cos(\varphi)&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}\mathbf{e}_\rho & \mathbf{e}_\varphi & \mathbf{e}_z\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}\sin(\vartheta)&0&\cos(\vartheta)\\0&1&0\\\cos(\vartheta)&0&-\sin(\vartheta)\end{pmatrix}$
	$\begin{pmatrix}\cos(\varphi)\sin(\vartheta)&\sin(\varphi)\sin(\vartheta)&\cos(\vartheta)\\-\sin(\varphi)&\cos(\varphi)&0\\\cos(\varphi)\cos(\vartheta)&\sin(\varphi)\cos(\vartheta)&-\sin(\vartheta)\end{pmatrix}$	$\begin{pmatrix}\sin(\vartheta)&0&\cos(\vartheta)\\0&1&0\\\cos(\vartheta)&0&-\sin(\vartheta)\end{pmatrix}$	$\begin{pmatrix}\mathbf{e}_r & \mathbf{e}_\varphi & \mathbf{e}_\vartheta\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$
Nabla			
∇f	$\begin{pmatrix}\frac{\partial f}{\partial x}\\\frac{\partial f}{\partial y}\\\frac{\partial f}{\partial z}\end{pmatrix}$	$\begin{pmatrix}\frac{\partial f}{\partial \rho}\\\frac{1}{\rho}\frac{\partial f}{\partial \varphi}\\\frac{\partial f}{\partial z}\end{pmatrix}$	$\begin{pmatrix}\frac{\partial f}{\partial r}\\\frac{1}{r\sin(\vartheta)}\frac{\partial f}{\partial \varphi}\\\frac{1}{r}\frac{\partial f}{\partial \vartheta}\end{pmatrix}$
$\nabla\cdot\mathbf{f}$	$\frac{\partial f_x}{\partial x}+\frac{\partial f_y}{\partial y}+\frac{\partial f_z}{\partial z}$	$\frac{1}{\rho}\frac{\partial(\rho f_\rho)}{\partial \rho}+\frac{1}{\rho}\frac{\partial f_\varphi}{\partial \varphi}+\frac{\partial f_z}{\partial z}$	$\frac{1}{r^2}\frac{\partial(r^2 f_r)}{\partial r}+\frac{1}{r\sin(\vartheta)}\frac{\partial f_\varphi}{\partial \varphi}+\frac{1}{r\sin(\vartheta)}\frac{\partial(\sin(\vartheta)f_\vartheta)}{\partial \vartheta}$
$\nabla\times\mathbf{f}$	$\begin{pmatrix}\frac{\partial f_z}{\partial y}-\frac{\partial f_y}{\partial z}\\\frac{\partial f_x}{\partial z}-\frac{\partial f_z}{\partial x}\\\frac{\partial f_y}{\partial x}-\frac{\partial f_x}{\partial y}\end{pmatrix}$	$\begin{pmatrix}\frac{1}{\rho}\frac{\partial f_z}{\partial \varphi}-\frac{\partial f_\varphi}{\partial z}\\\frac{\partial f_\rho}{\partial z}-\frac{\partial f_z}{\partial \rho}\\\frac{1}{\rho}\left(\frac{\partial(\rho f_\varphi)}{\partial \rho}-\frac{\partial f_\rho}{\partial \varphi}\right)\end{pmatrix}$	$\begin{pmatrix}\frac{1}{r\sin(\vartheta)}\left(\frac{\partial(\sin(\vartheta)f_\varphi)}{\partial \vartheta}-\frac{\partial f_\vartheta}{\partial \varphi}\right)\\\frac{1}{r}\left(\frac{\partial(r f_\vartheta)}{\partial r}-\frac{\partial f_r}{\partial \vartheta}\right)\\\frac{1}{r}\left(\frac{1}{\sin(\vartheta)}\frac{\partial f_r}{\partial \varphi}-\frac{\partial(r f_\varphi)}{\partial r}\right)\end{pmatrix}$
$\nabla^2\mathbf{f}$	$\frac{\partial^2\mathbf{f}}{\partial x^2}+\frac{\partial^2\mathbf{f}}{\partial y^2}+\frac{\partial^2\mathbf{f}}{\partial z^2}$	$\frac{\partial^2\mathbf{f}}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial\mathbf{f}}{\partial \rho}+\frac{1}{\rho^2}\frac{\partial^2\mathbf{f}}{\partial \varphi^2}+\frac{\partial^2\mathbf{f}}{\partial z^2}$	$\frac{\partial^2\mathbf{f}}{\partial r^2}+\frac{2}{r}\frac{\partial\mathbf{f}}{\partial r}+\frac{1}{r^2\sin(\vartheta)^2}\frac{\partial^2\mathbf{f}}{\partial \varphi^2}+\frac{1}{r^2\sin(\vartheta)}\frac{\partial\left(\sin(\vartheta)\frac{\partial\mathbf{f}}{\partial\vartheta}\right)}{\partial \vartheta}$

Kurvendiskussion

Tangenteneinheitsvektor

$\mathbf{T}(x_0) = \frac{\frac{d\mathbf{f}}{dx}(x_0)}{\left|\frac{d\mathbf{f}}{dx}(x_0)\right|}$

Hauptnormalenvektor

$\mathbf{N}(x_0) = \frac{\frac{d\mathbf{T}}{dx}(x_0)}{\left|\frac{d\mathbf{T}}{dx}(x_0)\right|}$

Binormalenvektor

$\mathbf{B}(x_0) = \mathbf{T}(x_0) \times \mathbf{N}(x_0)$

Krümmung

$\kappa(x) = \frac{|d\mathbf{T}|}{ds}$

Torsion

$\tau(x) = -\mathbf{N}\frac{d\mathbf{B}}{ds}$

Bogenlänge

$\left(\frac{ds}{dx}\right)^2 = \sum_{i=1}^{m} \left(\frac{df_i}{dx}(x)\right)^2$

Bahngeschwindigkeit

$\frac{d\mathbf{f}}{dx}(x) = \frac{ds(x)}{dx}\mathbf{T}(x)$

Beschleunigung

$\frac{d^2\mathbf{f}}{dx^2}(x) = \frac{d^2 s}{dx^2}(x)\mathbf{T}(x) + \left(\frac{ds}{dx}(x)\right)^2 \kappa(x)\mathbf{N}(x)$

Tangente

$\mathbf{t}(x_0, \lambda) = \mathbf{f}(x_0) + \lambda\mathbf{T}(x_0)$

Normale

$\mathbf{n}(x_0, \lambda) = \mathbf{f}(x_0) + \lambda\mathbf{N}(x_0)$

Hessematrix (symmetrisch)

$\mathbf{H}(f(\mathbf{x})) = \begin{pmatrix} f_{x_1 x_1} & \dots & f_{x_1 x_n} \\ \dots & \dots & \dots \\ f_{x_n x_1} & \dots & f_{x_n x_n} \end{pmatrix}$

Richtungsableitung

$\frac{\partial f}{\partial \mathbf{a}}(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \frac{\mathbf{a}}{|\mathbf{a}|}$

Tangentialebene

$(\mathbf{x} - \mathbf{x}_0) \cdot \nabla f(\mathbf{x}_0) = 0$

Extremum

$\Rightarrow \quad df(\mathbf{x}_0) = 0 \quad \Leftrightarrow \quad \nabla f(\mathbf{x}_0) = \mathbf{0}$

$\mathbf{H}(f(\mathbf{x}))$ positiv (negativ) definit $\Rightarrow$ Minimum (Maximum)
indefinit $\Rightarrow$ Sattelpunkt
sonst $\Rightarrow$ keine Aussage

Fehlerrechnung

$|\Delta f(\mathbf{x})| \leq \sum_j \left|\frac{\partial f}{\partial x_j}(\mathbf{x})\right| \Delta x_j$

x = Mittelwerte
Δx_j = maximaler absoluter Fehler

Integrale

Parameteranhängige Integrale (Leibnitzsche Regel)

$$\frac{d}{d\lambda} \int_{a(\lambda)}^{b(\lambda)} f(x,\lambda)dx = \int_{a(\lambda)}^{b(\lambda)} \frac{df}{d\lambda}(x,\lambda)dx + f(b(\lambda),\lambda)\frac{db}{d\lambda}(\lambda) - f(a(\lambda),\lambda)\frac{da}{d\lambda}(\lambda)$$

Kurvenintegral 1. Art $\Leftrightarrow$ 2. Art

$$\int_S f(\mathbf{x})dx = \int_s f(\mathbf{x}(t))\,|\dot{\mathbf{x}}(t)|\,dt = \int_s \left(f(\mathbf{x}(t))\frac{\dot{\mathbf{x}}(t)}{|\dot{\mathbf{x}}(t)|}\right) \cdot (\dot{\mathbf{x}}(t)dt) = \int_s \mathbf{v}(\mathbf{x}(t)) \cdot \dot{\mathbf{x}}(t)dt = \int_{\mathbf{S}} \mathbf{v}(\mathbf{x}) \cdot d\mathbf{x}$$

Flächenintegral 1. Art $\Leftrightarrow$ 2. Art

$$\iint_A f(\mathbf{x})dA = \iint_a f(\mathbf{x}(s,t))\,|\mathbf{x}_s \times \mathbf{x}_t|\,ds\,dt = \iint_a \left(f(\mathbf{x}(s,t))\frac{\mathbf{x}_s \times \mathbf{x}_t}{|\mathbf{x}_s \times \mathbf{x}_t|}\right) \cdot ((\mathbf{x}_s \times \mathbf{x}_t)\,ds\,dt) = \iint_a \mathbf{v}(\mathbf{x}(s,t)) \cdot (\mathbf{x}_s \times \mathbf{x}_t)\,ds\,dt = \iint_{\mathbf{A}} \mathbf{v}(\mathbf{x}) \cdot d\mathbf{A}$$

Anwendungen

Rotationskörper $A = 2\pi \int_0^s y\,ds = 2\pi y_s s$ $\quad V = \pi \int_a^b y^2 dx = 2\pi y_s A$

Statisches Moment $M = \sum_i m_i d_i$ $\quad M_x = \int_0^s y\,ds = \frac{1}{2}\int_a^b y^2\,dx$ $\quad M_y = \int_0^s x\,ds = \int_a^b xy\,dx$

Schwerpunkt $S = (x_s, y_s) = \left(\frac{1}{s}\int_0^s x\,ds, \frac{1}{s}\int_0^s y\,ds\right) = \left(\frac{1}{A}\int_a^b xy\,dx, \frac{1}{2A}\int_a^b y^2\,dx\right)$

Interpolation

Lagrange

$P_n(x) = \sum_{i=1}^{n} y_i \prod_{j=1}^{n} \frac{x-x_j}{x_i-x_j}$

Newton

$P_n(x) = \sum_{i=1}^{n} \alpha_i \prod_{j=1}^{n} \left(x - x_j\right)$

Trapezformel

$\int_a^b f(x)\mathrm{d}x = h\left(\frac{y_0+y_n}{2} + \sum_{i=1}^{n-1} y_i\right)$

Simpsonsche Formel

$\int_a^b f(x)\mathrm{d}x = \frac{h}{3}\left(y_0 + y_n + 2\sum_{i=1}^{\frac{n-1}{2}} y_{2i} + 4\sum_{i=1}^{\frac{n-1}{2}} y_{2i+1}\right)$

Schrittweises Integrieren $y' = f(x,y)$

Euler-Verfahren

$y_{k\tau_1} = y_k + hf(x_k, y_k)$

Runge-Kutta-Verfahren

$y_{k\tau_1} = y_k + \frac{h}{6}\left(T_{1k} + 2T_{2k} + 2T_{3k} + T_{4k}\right)$

$T_{1k} = f\left(x_k, y_k\right)$ $\quad T_{2k} = f\left(x_k + \frac{h}{2}, y_k + \frac{h}{2}T_{1k}\right)$

$T_{3k} = f\left(x_k + \frac{h}{2}, y_k + \frac{h}{2}T_{2k}\right)$ $\quad T_{4k} = f\left(x_k + h, y_k + hT_{3k}\right)$

Differentialgelichungen

$y' = g(x)h(y)$	$y = y_0$ mit $h(y_0) = 0$	$\Rightarrow$	$\int \frac{1}{h(y)}\mathrm{d}y = \int g(x)\mathrm{d}x + c$	
$y' = f\left(\frac{y}{x}\right)$	$u = \frac{y}{x}$	$\Rightarrow$	$\int \frac{1}{f(u)-u}\mathrm{d}u = \ln\left(\lvert x\rvert\right) + c$	
$F(x, y', y'') = 0$	$u = y'$	$\Rightarrow$	$F(x, u, u') = 0$	$y = \int u(x, c_1)\mathrm{d}x$
$F(y, y', y'') = 0$	$u(y) = y'$	$\Rightarrow$	$F(y, u, u'u) = 0$	$\int \frac{1}{u(y,c_1)}\mathrm{d}y = x + c_2$
$y' + p(x)y = q(x)y^n$	$u = y^{1-n}$	$\Rightarrow$	$u' + (1-n)p(x)u = (1-n)q(x)$	

Lineare Differentialgleichungen

$\sum_{i=0}^{n} f_i y^{(i)} = p(x) = Q_\alpha(x)\sin(\nu x + \beta)\exp(\mu x)$

$y = \sum_{i=1}^{n} c_i y_i + y_p$

Wronski-Determinante $\neq 0 \quad \Rightarrow \quad y_1 \dots y_n$ Fundamentalsystem

$$W(y_1, y_2, \dots, y_n) = \begin{vmatrix} y_1 & \dots & y_n \\ \dots & \dots & \dots \\ y_1^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix} \neq 0$$

Inhomogen: Variation der Konstanten

$$\begin{pmatrix} y_1 & \dots & y_n \\ \dots & \dots & \dots \\ y_1^{(n-1)} & \dots & y_n^{(n-1)} \end{pmatrix} \cdot \begin{pmatrix} c_1' \\ \vdots \\ c_n' \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ p(x) \end{pmatrix}$$

$f_i = \mathrm{const}$

Charakteristisches Polynom $\quad P(\lambda) = \sum_{i=0}^{n} f_i \lambda^{(i)} = 0 = \prod_{j=0}^{m\leq n}\left(\lambda - \lambda_j\right)^{k_j}$

$y = \sum_{j=0}^{m}\sum_{i=0}^{k_j-1} c_{ji} x^i \exp(\lambda_j x)$ $\qquad y_p = R_\alpha(x)x^k\exp(\lambda x) = \left(S_\alpha(x)\sin(\nu x) + T_\alpha(x)\cos(\nu x)\right)x^k\exp(\mu x)$